METHODS OF APPLIED MATHEMATICS COMPREHENSIVE EXAMINATION AUGUST 2015

Work on as many of the following problems as possible. Turn in *all* your work.

(1) Consider two bodies of mass m_1 and m_2, respectively, joined by a spring of constant k, in collinear motion along the z-axis of a cartesian frame in $\mathbb{R}^3$. If the position of the first body is at $z = r_1(t)$ and the second is at $z = r_2(t)$, write down the evolution equations given by Newton's second law for these functions of time t, when both bodies are immersed in a gravitational force field $F = -GM_e m_j/r^2$, $j = 1, 2$, where r is the distance between the j-th body and the Earth's center, M_e are the Earth's and body's mass, respectively, and G is the universal gravitational constant.

(a) Non-dimensionalize the equations of motion; assume $\epsilon \equiv m_2/m_1 \ll 1$, with $M_e \gg m_1$, and that the initial separation h between the two bodies is $0 < h \equiv r_1(0) - r_2(0)$, with $h \ll r_1(0)$.

(b) Neglecting gravity, find the solution of these equations corresponding to zero initial velocities.

(c) Write an asymptotic expansion of the equations of motion, and find the leading order terms of their solutions in the absence of gravity.

(d) Assuming that the Earth is shrunk to a point at the origin and the gravitational interaction between m_1 and m_2 is negligible, find the leading order terms of the motion equations when gravity is included, and sketch the leading order solutions noting their time scale of validity.

(2) Consider the following function of the complex variable z in the complex plane

$$f(z) = \frac{1}{\sqrt{z^2-1}} \operatorname{atan}\left(\frac{1}{\sqrt{z^2-1}}\right) :$$

(a) Classify all singularities and propose branch cuts, if necessary, to make the function single valued on appropriate domains.

(b) Discuss the convergence of the real integral

$$\int_{-1}^{1} \frac{1}{\sqrt{|a^2-t^2|}} \operatorname{atan}\left(\frac{1}{\sqrt{1-t^2}}\right) dt$$

as a function of the real parameter a.

(c) Propose a strategy for computing the real integral when $a = 1$ based on the study of the function $f(z)$. Discuss your proposal even if you cannot carry out all the steps to evaluate the value of the integral, if it is finite.

(3) Consider the rapidly varying diffusivity:

$$K(x, y; \epsilon) = A + F(x/\epsilon^2) + G(y/\epsilon)$$

where A is chosen to guarantee K is positive, and ϵ is a small constant. By applying iterated homogenization, average the following diffusion equation subject to the rapidly varying "potential" $V(x/\varepsilon^2) > 0$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(x, y; \epsilon)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K(x, y; \epsilon)\frac{\partial u}{\partial y}\right) + V\left(\frac{x}{\varepsilon^2}\right) u$$

$$u(x, y, 0) = u_0(x, y),$$

by computing a leading order effective equation governing the evolution as $\epsilon \to 0$ over the (x, y)-plane, assuming the functions $F(x)$, $V(x)$ and $G(y)$ are mean zero, periodic, and share the same period. Solve the averaged equation in free space.

(4) Consider the 4th-order ordinary differential equation on the real line $x \in \mathbb{R}$,

$$\frac{d^4y}{dx^4} = xy\,.$$

(a) By using a contour integral representation in the appropriate complex Laplace-image plane, discuss whether solutions decaying as $|x| \to \infty$ exist.
(b) Find the leading order asymptotic expansion of one of these solutions as $x \to \infty$ by appropriate deformations of the contour integral representation.

(5) Consider the eigenvalue problem on the real line $x \in \mathbb{R}$

$$\epsilon y'' + (\cos^2(x) + \lambda)y = 0, \qquad y(x) < \infty, \text{ as } \quad |x| \to \infty.$$

(a) Identify the range of λ for bounded eigenfunctions to exist.
(b) As $\epsilon \to 0$ compute acceptable solutions by WKBJ approach in this limit.

(6) Find two term asymptotic expansions as $\epsilon \to 0$ for all roots of the equation:

$$\epsilon z^4 + z^2 - 2z + 1 = \epsilon$$

(7) Consider the following initial value problem for the equation

$$u_t + (\varepsilon + x)u_x = \epsilon u u_x\,, \qquad u(x,0) = u_0(x)\,,$$

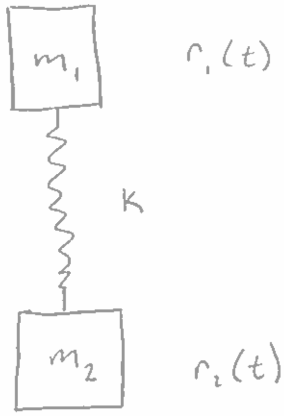
where ϵ is a fixed nonzero small real parameter.
(a) Solve with the method of characteristics for $\epsilon = 0$ and all times $t \geq 0$ on the real line x assuming the initial condition $u_0(x)$ is a "nice" function decaying as $|x| \to \infty$.
(b) Set up a regular perturbation expansion for $u(\cdot, \cdot, \epsilon)$ and find the first correction to the leading order solution found above.

(8) (a) Explain the difference between pointwise convergence and asymptotic convergence. Illustrate with the particular example of power series.
(b) Consider the map that associates to a function $f(z)$ its asymptotic expansion as $z \to z_0$, where z_0 is a point of the complex plane, including infinity. Thus $f(z) \sim g_N(z)$ where

$$g_N(z) \equiv \sum_{n=0}^{N} a_n\phi_n(z)\,, \quad \text{with} \quad \lim_{z\to z_0} \left|\frac{f(z) - g_N(z)}{\phi_N(z))}\right| = 0\,,$$

for some sequence of order functions $\phi_n(z)$, $n = 0, \ldots, N$. Is this mapping invertible? Discuss and illustrate with examples.
(c) Is the mapping associative? Is it distributive? Discuss.

Problem 1



$$\begin{cases} m_1 \ddot{r}_1 = -\dfrac{GMm_1}{|r_1 - r_2|^2} - K(r_1 - r_2 - d) & (1) \\[2ex] m_2 \ddot{r}_2 = -\dfrac{GMm_2}{|r_1 - r_2|^2} + K(r_1 - r_2 - d) & (2) \end{cases}$$

where d = length of spring at rest

a) Let $L = d$ and $T = \sqrt{\frac{m_2}{K}}$ and define

$$r = zL \quad \text{and} \quad t = \tau T.$$

$$\underbrace{\qquad\qquad\qquad}_{\text{non-dim}}$$

$$\Rightarrow \frac{dr}{dt} = \frac{d(zL)}{dt} = \frac{L}{T}\frac{dz}{d\tau} = \frac{L}{T}\dot{z}$$

Let's non-dim eqn (1):

$$m_1 \frac{L}{T^2} \ddot{z}_1 = -\frac{GMm_1}{L^2(z - z_2)^2} - KL(z_1 - z_2 - 1)$$

$$\ddot{z}_1 = -\frac{GMT^2}{L^3(z - z_2)^2} - \frac{KT^2}{m_1}(z_1 - z_2 - 1) \qquad \text{Let } \varepsilon = \frac{m_2}{m_1} \ll$$

$$\begin{cases} \ddot{z}_1 = -\dfrac{GMT^2}{L^3(z - z_2)^2} - K\varepsilon(z_1 - z_2 - 1) & (3) \\[2ex] \ddot{z}_2 = -\dfrac{GMT^2}{L^3(z - z_2)^2} + K(z_1 - z_2 - 1) & (4) \end{cases}$$

b) Neglecting gravity:

$$\begin{cases} \ddot{z}_1 = -K\varepsilon(z_1 - z_2 - 1) & (5) \\ \ddot{z}_2 = K(z_1 - z_2 - 1) \quad (6) & \Rightarrow z_1 = \frac{\ddot{z}_2}{K} + z_2 + 1 \quad (7) \\ \dot{z}_i(0) = 0 & \end{cases}$$

$$\Rightarrow \frac{\ddddot{z}_2}{K} + \ddot{z}_2 = -K\varepsilon\left(\frac{\ddot{z}_2}{\cancel{K}} + z_2 + 1 - \cancel{z_2} - \cancel{1}\right)$$

Ansantz: $z_2 = \bar{Z} + \varepsilon \bar{Z}_1 + \varepsilon^2 \bar{Z}_2 + \ldots$

$O(1)$: $\ddddot{\bar{Z}} + K\ddot{\bar{Z}} = 0$ Let $u = \ddot{\bar{Z}}$

$$\ddot{u} = -Ku$$

$$u = K(\sin\tau + \cos\tau)$$

$$\ddot{\bar{Z}} = K(\sin\tau + \cos\tau)$$

$$\dot{\bar{Z}} = K(-\cos\tau + \sin\tau) + a \qquad \dot{\bar{Z}}(0) = -K + a = 0$$
$$a = K$$

$$\bar{Z} = K(-\sin\tau - \cos\tau) + Kt + \bar{Z}(0)$$

$$z_2 \sim K(-\sin\tau - \cos\tau) + Kt + z_2(0) \quad \text{as } \varepsilon \to 0$$

z_1 can be found using eqn (7)

c) This can be done through a process simillar to the one above but including the gravity term. By shrinking the Earth to a point, we can use $z_e \sim 0$ and it would be convinient to use $\frac{1}{z^2} = \sum_{n=0}^{\infty} (z^2+1)^n$.

Problem 2

$$f(z) = \frac{1}{\sqrt{z^2-1}} \operatorname{atan}\left(\frac{1}{\sqrt{z^2-1}}\right)$$

a) Branch points:

$\sqrt{\ }$: $z^2 - 1 = 0$

$z = \pm 1$

Note: $z = \pm 1$ are also essential singularities

$$\operatorname{atan}(x) = \frac{1}{2i} \log\left(\frac{1-xi}{1+xi}\right)$$

BP's for log at 0 and ∞

∞: $1 + xi = 0$	0: $1 - xi = 0$
$x = i$	$x = -i$
$z^2 - 1 = -1$	$z^2 - 1 = 1$
$z = 0$	$z = 0$

Check if $z = 0$ is a bp

$$\log\Big(\underbrace{1 - \sqrt{\rho_1\rho_2}\, e^{\frac{\theta_1+\theta_2}{2}}}_{1}\Big) - \log\Big(\underbrace{1 + \sqrt{\rho_1\rho_2}\, e^{\frac{\theta_1+\theta_2}{2}}}_{2}\Big) \overset{?}{=} \log\Big(\overbrace{1 - \sqrt{\rho_1\rho_2}\, e^{\frac{\theta_1-\theta_2}{2} + 2\pi}}^{3}\Big) - \log\Big(\underbrace{1 - \sqrt{\rho_1\rho_2}\, e^{\frac{\theta_1-\theta_2}{2} + 2\pi}}_{4}\Big)$$

$$\ln(\text{stuff}) + \arg(1) - \arg(2) = \ln(\text{stuff}) + \arg(3) - \arg(4)$$

$$\phi_1 - \phi_2 = \phi_1 + \cancel{2\pi} - \phi_2 - \cancel{2\pi}$$

So $z = 0$ is not a bp.

Let's also check if ∞ is a bp

$z = \frac{1}{s}$

$$\Rightarrow \quad \frac{s}{\sqrt{1-s^2}} \operatorname{atan}\left(\frac{s}{\sqrt{1-s^2}}\right) \longrightarrow 0 \operatorname{atan}(0) = 0$$

as $s \to 0$

So ∞ is not a bp either.

Thus our only branch points are $z = \pm 1$

Proposed branch cuts:

Problem 3

Aug 2015

$$K(x,y,\varepsilon) = A + F\left(\tfrac{x}{\varepsilon^2}\right) + G\left(\tfrac{y}{\varepsilon}\right)$$

$$u_t = \partial_x(K\,\partial_x u) + \partial_y(K\,\partial_y u) + V\left(\tfrac{x}{\varepsilon^2}\right)u$$

Let $w = \frac{x}{\varepsilon^2}$ and $z = \frac{y}{\varepsilon}$ $\Rightarrow$ $\partial_x \mapsto \partial_x + \frac{1}{\varepsilon^2}\partial_w$ & $\partial_y \mapsto \partial_y + \frac{1}{\varepsilon}\partial_z$

$$\Rightarrow \quad u_t = \left(\partial_x + \tfrac{1}{\varepsilon^2}\partial_w\right)\left[K\left(\partial_x + \tfrac{1}{\varepsilon^2}\partial_w\right)u\right] + \left(\partial_y + \tfrac{1}{\varepsilon}\partial_z\right)\left[K\left(\partial_y + \tfrac{1}{\varepsilon}\partial_z\right)\right] + V(w)\cdot u$$

Ansatz: $u = \bar{u} + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$

$\mathcal{O}\left(\frac{1}{\varepsilon^4}\right)$: $\partial_w(K\partial_w \bar{u}) = 0 \Rightarrow u(x,y,z,t)$

$\mathcal{O}\left(\frac{1}{\varepsilon^3}\right)$: $\partial_w(K\partial_w u_1) = 0 \Rightarrow u_1(x,y,z,t)$

$\mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$: $\partial_w(K\partial_w u_2) + \cancel{\partial_x(K\partial_w \bar{u})}^{\to 0} + \partial_w(K\partial_x \bar{u}) + \partial_z(K\partial_z \bar{u}) = 0$

$$\langle \partial_w(K\partial_w u_2 + K\partial_x \bar{u})\rangle_w + \cancel{\langle \partial_z(K\partial_z \bar{u})\rangle_w}^{\to 0} = 0$$

periodic   indep w

$$K(\partial_w u_2 + \partial_x \bar{u}) = A$$

$$\langle \partial_w u_2 + \partial_x \bar{u}\rangle_w = \left\langle \frac{A}{K}\right\rangle_w$$

$$\partial_x \bar{u} = A\left(\langle K\rangle_w^h\right)^{-1}$$

$$K(\partial_w u_2 + \partial_x \bar{u}) = \langle K\rangle_w^h\,\partial_x \bar{u} \quad \Rightarrow \quad \bar{u}(x,y,t)$$

$\mathcal{O}\left(\frac{1}{\varepsilon}\right)$: $\partial_w(K\partial_w u_3) + \cancel{\partial_x(K\partial_w u_1)}^{\to 0} + \partial_w(K\partial_x u_1) + \partial_z(K\partial_z u_1) + \cancel{\partial_y(K\partial_z \bar{u})}^{\to 0} + \partial_z(K\partial_y \bar{u}) = 0$

$$\cancel{\langle \partial_w(K\partial_w u_3 + K\partial_x u_1\rangle_w}^{\to 0} + \langle \partial_z(K\partial_z u_1 + K\partial_y \bar{u}\rangle_w = 0$$

$$\langle K\rangle_w(\partial_z u_1 + \partial_y \bar{u}) = B$$

$$\partial_y \bar{u} = B\left(\langle\langle K\rangle_w\rangle_z^h\right)^{-1}$$

$$\langle K\rangle_w(\partial_z u_1 + \partial_y \bar{u}) = \langle\langle K\rangle_w\rangle_z^h\,\partial_y \bar{u}$$

$$O(1): \partial_w(K\partial_w u_4) + \partial_x(K\partial_w u_2) + \partial_w(K\partial_x u_2) + \partial_z(K\partial_z u_2) + \partial_z(K\partial_y u_1) + \partial_y(K\partial_z u_1)$$

$$+ v(z)\bar{u} + \partial_x(K\partial_x\bar{u}) + \partial_y(K\partial_y\bar{u}) = u_t$$

$$\langle \partial_w(\underbrace{\cancel{K\partial_w u_4 + K\partial_x u_2}}_{FA})\rangle_w \to 0 + \langle \partial_z(K\partial_z u_2 + K\partial_y u_1)\rangle_w + \langle \partial_x(\underbrace{K\partial_w u_2 + K\partial_x\bar{u}}_{\langle K\rangle_w^h \bar{u}_{xx}})\rangle_w$$

$$+ \langle \partial_y(K\partial_z u_1 + K\partial_y\bar{u})\rangle_w + \langle v(z)\bar{u}\rangle_w = \langle \bar{u}_t\rangle_w$$

$$\langle \cancel{v(z)}\bar{u}\rangle_z + \langle \partial_z\cancel{\langle K\partial_z u_2 + K\partial_y u_1\rangle_w}\rangle_z \to 0 + \langle\langle K\rangle_w^h \bar{u}_{xx}\rangle_z + \langle \partial_y(\underbrace{\langle K\rangle_w(\partial_z u_1 + \partial_y\bar{u})}_{\langle\langle K\rangle_w\rangle_z^h \bar{u}_y})\rangle_z = \langle\bar{u}_t\rangle_z$$

periodic

indep of z

$$\bar{u}_t = \underbrace{\langle\langle K\rangle_w^h\rangle_z}_{\alpha}\bar{u}_{xx} + \underbrace{\langle\langle K\rangle_w\rangle_z^h}_{\beta}\bar{u}_{yy}$$

$$\therefore \boxed{\bar{u} = \mathcal{F}\{e^{(\alpha x^2 + \beta\eta^2)t}\}}$$

where $\mathcal{F}(\bar{u}_{xx}) = x^2$

$\mathcal{F}(\bar{u}_{yy}) = \eta^2$

Problem 4

Aug 2015

$$y^{(4)} = xy \qquad (1)$$

a) For some path C:

$$y = \int_c \hat{y}\, e^{sx}\, ds \qquad (2)$$

$$\Rightarrow y^{(4)} = \int_c \hat{y}\, s^4 e^{sx}\, ds$$

$$\text{\&}\quad xy = \int_c \hat{y}\, e^{sx} x\, ds = \cancelto{0}{e^{sx}\hat{y}\Big|_c} - \int_c e^{sx}\, \hat{y}_s\, ds = -\int_c e^{sx}\, \hat{y}_s\, ds$$

$$u = \hat{y} \qquad v = e^{sx}$$

$$du = \hat{y}_s\, ds \qquad dv = x e^{sx}\, ds$$

<u>Conditions on C:</u>

i) conv. integrals

ii) $e^{sx}\hat{y}\Big|_c = 0$

Plug into (1):

$$\int_c s^4 e^{sx}\, \hat{y}_s\, ds = -\int e^{sx}\, \hat{y}\, ds$$

$$\int_c e^{sx}\left(s^4 \hat{y} + \hat{y}_s \right) ds = 0$$

$$s^4 \hat{y} + \hat{y}_s = 0$$

$$\hat{y}_s + s^4 \hat{y} = 0$$

$$\hat{y} = a\, e^{-\frac{s^5}{5}} \qquad \text{where} \quad s = re^{i\theta}$$

Plug into (2):

$$\boxed{y = a \int_c e^{sx - s^5/5}\, dx}$$

For this to decay as $x \to \infty$, we need $y \to 0$.

This will occur when $Re(s^5) > 0$ $\because$ s^5 will dominate s,

b) $s^5 = r^5 e^{i5\theta}$

$Re(s^5) > 0$ if $\cos 4\theta > 0$

$-\frac{\pi}{2} + 2\pi k < 5\theta < \frac{\pi}{2} + 2\pi k$ $\quad k = 0, 1, 2, 3, 4$

$-\frac{\pi}{10} + \frac{\pi k}{5} < \theta < \frac{\pi}{10} + \frac{\pi k}{5}$

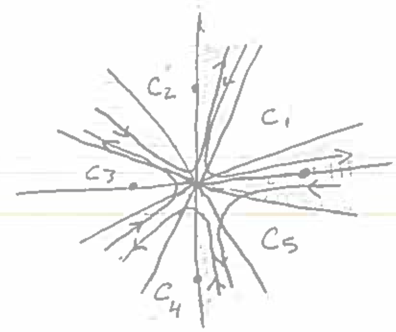
$k=0: \ -\frac{\pi}{10} < \theta < \frac{\pi}{10}$

$k=1: \ \frac{3\pi}{10} < \theta < \frac{5\pi}{10} = \frac{\pi}{2}$

$k=2: \ \frac{7\pi}{10} < \theta < \frac{9\pi}{10}$

$k=3: \ \frac{11\pi}{10} < \theta < \frac{13\pi}{10}$

$k=4: \ \frac{3\pi}{2} = \frac{15\pi}{10} < \theta < \frac{17\pi}{10}$



Choose C to be countor that goes to zero as $x \to \infty$ and easy to deform thru s.p.

$\Rightarrow C = C_3$

$$y = \int_{C_i} e^{-\frac{s^5}{5} + sx}\, ds = \int_{C_i} x^{\alpha} e^{-\frac{\xi^5 x^{5\alpha}}{5} + \xi x^{\alpha+1}}\, d\xi$$

$5\alpha = \alpha + 1$

$\alpha = 1/5$

$s = x^{\alpha}\xi$

$ds = x^{\alpha} d\xi$

$$= \int_{C_i} x^{1/5} e^{x^{5/4}(-\xi^5/4 + \xi)}\, d\xi$$

$h(\xi) = -\frac{\xi^5}{5} + \xi$

$h'(\xi) = -\xi^4 + 1 = 0 \qquad \xi_k = e^{\frac{k\pi}{2} i} \qquad k = 0, 1, 2, 3$

$h''(\xi) = -4\xi^3 \neq 0 \quad \forall \xi_k \quad \Rightarrow n = 2$

$h''(-1) = 4e^{i0} \Rightarrow \alpha = 0$

We need $\operatorname{Im}(h(z)) = 0$ and $\operatorname{Re}(h(z)) < 0$

so $\sin(\alpha + 2\theta_m) = 0$ and $\cos(x + 2\theta_m) < 0$ $m = 0, 1$

$$\alpha + 2\theta_m = m\pi$$
$$\theta_m = \frac{m\pi}{2}$$
$$\theta_0 = 0 \quad \theta_1 = \frac{\pi}{2}$$

θ_0: $\cos(0) = 1 \not< 0$

θ_1: $\cos(\pi) = -1 < 0$

$$\theta_{SD} = \frac{\pi}{2}$$

$$y = \int_{C_3} x^{1/4} e^{x^{5/4} h(\xi)} \, d\xi \sim x^{1/4} e^{x^{5/4} h(-1)} \sqrt{\frac{2\pi}{x^{5/4} |h''(-1)|}} \, e^{i\pi/2}$$

as $x \to \infty$

$\uparrow$ by Method of SD

$$\boxed{y(x) \sim x^{1/4} i e^{-4/5 x^{5/4}} \sqrt{\frac{\pi}{2x}} \quad \text{as} \quad x \to \infty}$$

Note: We should deform the C_3 curve to go through $\xi^* = -1$

Aug 2015

Problem 5

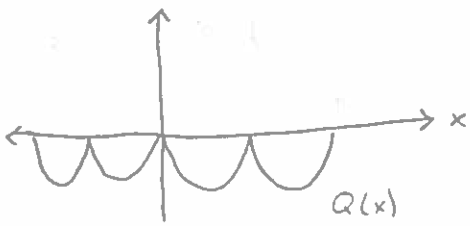
$$\varepsilon y'' + (\cos^2(x) + \lambda)\, y = 0$$

$$\varepsilon y'' - (-\cos^2(x) - \lambda)\, y = 0$$

$Q(x) = -\cos^2(x)$

$E = \lambda$

$$\varepsilon y'' - (\underbrace{Q(x) - E}_{U(x)})\, y = 0$$



a) $\min(Q(x)) < E < \max(Q(x))$

$$\boxed{-1 < \lambda < 0}$$

b) We know: $y \sim e^{\frac{S_0}{\varepsilon} + S_1 + \cdots}$

where $S_0 = \omega \int^x \sqrt{Q(t) - \lambda}\; dt$

$= \omega \int^x \sqrt{-\cos^2 t - \lambda}\; dt$

$= ?$

and $S_1 = \frac{1-2}{2(2)} \log(-\cos^2 t - \lambda)$

$= -\frac{1}{4} \log(-\cos^2(x) - \lambda)$

$= ?$

Given more time, I'd explore various integration techniques for S_1 and maybe try a Taylor series expansion for S_1.

Aug 2015

Problem 6

$$\varepsilon z^4 + z^2 - 2z + 1 = \varepsilon \qquad (1)$$

Consider the unpreturbed problem:

$$z^2 - 2z + 1 = 0$$

$$(z-1)^2 = 0$$

$z = 1$ Non-degen $\because$ $\varepsilon + 1 - 2 + 1 = \varepsilon$ ✓

$\Rightarrow$ $z_2 = 1 + \varepsilon^{1/2} a_1 + \dots$

Plug into (1):

$$\varepsilon(1 + \varepsilon^{1/2} a_1 + \dots)^4 + (\cancel{1} + \varepsilon^{1/2} a_1 + \dots - \cancel{1})^2 = \varepsilon$$

$O(\varepsilon)$: $1 + a_1^2 = 1$

$a_1 = 0$

$O(\varepsilon^2)$: $4a_2 + a_2^2 = 0$

$\underline{a_2 = 0}$ or $\underline{a_2 = -1/4}$

$\uparrow$ corresponds to $z_1 = 1$ soln

$$\boxed{\begin{array}{l} z_1 \sim 1 + 0\varepsilon \\ z_2 \sim 1 - 1/4\,\varepsilon \end{array}}$$ as $\varepsilon \to 0$

Let $x = \varepsilon^{\alpha} z$ $\Rightarrow$ $z = \varepsilon^{-\alpha} x$ and plug into (1):

$$\varepsilon^{1-4\alpha} x^4 + \varepsilon^{-2\alpha} x^2 - 2\varepsilon^{-\alpha} x + 1 = \varepsilon$$

$1 - 4\alpha = -2\alpha$

$1 = 2\alpha$

$\alpha = 1/2$

$$\varepsilon^{-1} x^4 + \varepsilon^{-1} x^2 - 2\varepsilon^{-1/2} x + 1 = \varepsilon$$

$$x^4 + x^2 - 2x\varepsilon^{1/2} + \varepsilon = \varepsilon^2 \qquad (2)$$

Consider the unpreturbed problem:

$$x^4 + x^2 = 0$$

$$x^2(x^2+1) = 0$$

$$x = 0, 0, -i, i$$

correspond with z_1 and z_2

$$x_3 = -i + b_1 \varepsilon^{1/2} + \dots$$

$$x_4 = i + c_1 \varepsilon^{1/2} + \dots$$

Plug into (2):

$$(-i + b_1\varepsilon^{1/2})^4 + (-i + b_1\varepsilon^{1/2})^2 - 2\varepsilon^{1/2}(-i + b_1\varepsilon^{1/2}) + \varepsilon = \varepsilon^2$$

$$O(\varepsilon^{1/2}): \; -4ib_1 - 2ib_1 + 2i = 0$$

$$b_1 = -1$$

Similarly, $c_1 = -1$

$$\Rightarrow \; x_3 = -1 - \varepsilon^{1/2}$$

$$x_4 = 1 - \varepsilon^{1/2}$$

Recall $z = \varepsilon^{-1/2} x$

$$\Rightarrow \; \boxed{\begin{aligned} z_3 &\sim -\frac{1}{\sqrt{\varepsilon}} - 1 \\ z_4 &\sim \frac{1}{\sqrt{\varepsilon}} - 1 \end{aligned}} \quad \text{as } \varepsilon \to 0$$

<u>Problem 7</u> Aug 2015

$$u_t + (\varepsilon + x)u_x = \varepsilon u u_x \qquad (1)$$

a) Let $\varepsilon = 0 \Rightarrow u_t + x u_x = 0 \qquad (2)$

Define $\begin{cases} z(s) = u(X(s), T(s)) \\ X(0) = x \\ T(0) = t \end{cases}$

From coeff's of (2) we know:

$$\begin{cases} \frac{dX}{ds} = X(s) \\ \frac{dT}{ds} = 1 \end{cases} \Rightarrow \begin{cases} X(s) = c_1 e^s \\ T(s) = s + c_2 \end{cases} \Rightarrow \begin{cases} X(s) = x e^s \\ T(s) = s + t \end{cases}$$

$$\frac{dz}{ds} = 0 \Rightarrow z(s) = z(0)$$

$\begin{cases} z(s) = u(X(s), T(s)) \\ z(0) = u(x,t) \end{cases}$ choose $s = -t$

$$z(0) = z(-t)$$

$$\therefore \boxed{u(x,t) = u_0(xe^{-t})}$$

b) Anzats: $u(x,t) = \bar{u}(x,t) + \varepsilon u_1(x,t) + \ldots$

where $\bar{u}(x,t) = u_0(x)$

$u_i(x,0) = 0 \quad \forall i \geq 1$

$\Theta(1)$: $\bar{u}_t + x\bar{u}_x = 0$

$\Rightarrow \bar{u}(x,t) = u_0(xe^{-t})$ by part (a)

$\Theta(\varepsilon)$: $u_{1_t} + \bar{u}_x + xu_{1x} = \bar{u}\bar{u}_x$

$$\begin{cases} z(s) = u_1(X(s), T(s)) \\ X(s) = x \\ T(s) = 1 \end{cases} \Rightarrow \begin{array}{l} X(s) = xe^s \\ T(s) = s+t \end{array}$$

$$\frac{dz}{ds} = \bar{u}\bar{u}_x - \bar{u}_x = g(X(s), T(s))$$

$$\Rightarrow z(s) = z(0) + \int_0^s g(X(v), T(v))\,dv = u_1(X(s), T(s))$$

Choose $s = -t$

$$z(-t) = z(0) + \int_0^{-t} g(X(v), T(v))\,dv = u_1(xe^{-t}, 0) = 0$$

where $z(0) = u_1(x,t)$. Solve above for u_1:

$$\therefore \boxed{u_1(x,t) = -\int_0^{-t} g(xe^v, v+t)\,dv}$$

Problem 8 Aug 2015

a) Let $\Omega \subset \mathbb{C}$, $x_0 \in \Omega$ and f_n be a sequence of funcs.

Ptws: We say $f_n \to f$ conv ptws if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ st $\forall n > N$ we get $|f_n(x_0) - f(x_0)| < \varepsilon$.

Note: We are only concerned with the fixed pt x_0 and N can vary as ε changes.

Asymp: Fix N. We say $f_N \to f$ conv asymp'ly if $\forall \varepsilon > 0$ $\exists \delta > 0$ st $|x - x_0| < \delta \Rightarrow |f_N(x) - f(x)| < \varepsilon$.

Note: We are only concerned w/ the nbhd of x_0, not x_0 itself and N is fixed.

Ex: In each example, I'll use Ratio test to show conv.

i) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \left| \frac{x}{n+1} \right| \xrightarrow[n \to \infty]{x \text{ fixed}} 0 \quad \therefore \text{ ptws conv}$$

$$\xrightarrow[n \text{ fixed}]{x \to \infty} \infty \quad \therefore \text{ not asymp conv}$$

$\sum_{n=0}^{\infty} \frac{n!}{x^n}$

$$\left| \frac{(n+1)!}{x^{n+1}} \cdot \frac{x^n}{n!} \right| = \left| \frac{n+1}{x} \right| \xrightarrow[n \to \infty]{x \text{ fixed}} \infty \quad \therefore \text{ not ptw conv}$$

$$\xrightarrow[n \text{ fixed}]{x \to \infty} 0 \quad \therefore \text{ asymp conv}$$

b) ?

c) Distributive

Sps $f \sim \tilde{f}$ and $g \sim \tilde{g}$ as $z \to z_0$ for both. Then yes,

$$f + g \sim \tilde{f} + \tilde{g}$$

so $\sim$ is distributive.

Now sps we also have $h \sim \tilde{h}$ as $z \longrightarrow z_0$.
Making use of distribution...

$$\Rightarrow (f+g)+h \sim \widetilde{(f+g)} + \tilde{h} \sim (\tilde{f} + \tilde{g}) + \tilde{h} = \tilde{f} + \tilde{g} + \tilde{h}$$

$$f + (g+h) \sim \tilde{f} + \widetilde{(g+h)} \sim \tilde{f} + (\tilde{g} + \tilde{h}) = \tilde{f} + \tilde{g} + \tilde{h}$$

$\because$ + is associtive

Thus $\sim$ is assosiative.