6/26

§2.2 Limits of Funcs

From precalc: $x \to a^{+/-}$ then $f(x) = ?$. These were informal limit

We should write $\lim_{x \to a^{+/-}} f(x) = ?$.

$\lim_{x \to a^+} f(x) =$ limit as x get's close to a from the right

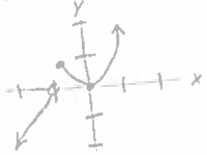
$\lim_{x \to a^-} f(x) =$ " " left

Limit means "What happens to f(x) as x get's close NOT x = "

F Ex1: $f(x) = \begin{cases} 1+x & x < -1 \quad \text{(left)} \\ x^2 & x \geq -1 \quad \text{(right)} \end{cases}$

Find the left and right handed limits.

Method 1: Graph



Time consuming

Method2: Table

Left

x	y
-1.1	-0.1
-1.01	-0.01
-1.001	-0.001
~-1	~0

Right

x	y
-0.9	.81
-0.99	.9801
-0.999	.9980
~-1	~1

Tedious

Method3: Plug in

$1 + (-1) = 0$

$(1)^2 = 1$

Quick

Ans: $\lim_{x \to -1^+} f(x) = 1$

$\lim_{x \to -1^-} f(x) = 0$

Defn: If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$ then the limit of f exists

and $\lim_{x \to a} f(x) = \lim_{x \to a^{+/-}} f(x)$.

Ex 2: Does $\lim_{x \to -1} f(x)$ exist in Ex 1?

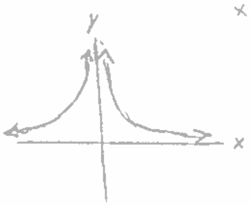
No

Ex 3: Find $\lim_{x \to 1} \frac{x-1}{x^2-1}$.

$$= \lim_{x \to 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}$$

I like this ∵ f(1) DNE, limits help us a way to think about what "happens" at x=1,

Ex 4: Find $\lim_{x \to 0} \frac{1}{x^2}$



Table

x	y
± 0.1	100
± 0.01	10000
± 0.001	1,000,000
	exploding

$\lim_{x \to 0} \frac{1}{x^2} = \infty$

You try

Ex 5: Find $\lim_{x \to 0} \frac{1}{x^3}$. DNE

Ex 6: Find $\lim_{x \to 3} \frac{2x}{x-3}$.

$\lim_{x \to 3^+} \frac{2x}{x-3} = \frac{+}{+} = \infty$

$\lim_{x \to 3^-} \frac{2x}{x-3} = \frac{+}{-} = -\infty$

§2.3 Limit Laws

Sps c is a constant. Then

1/2) $\lim\limits_{x\to a}(f(x)\pm g(x)) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

3) $\lim\limits_{x\to a} c\cdot f(x) = c\cdot\lim\limits_{x\to a} f(x)$

4) $\lim\limits_{x\to a} f(x)g(x) = (\lim\limits_{x\to a} f(x))(\lim\limits_{x\to a} g(x))$

5) $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ where $\lim\limits_{x\to a} g(x)\neq 0$

6) $\lim\limits_{x\to a}(f(x))^n = (\lim\limits_{x\to a} f(x))^n$ $n\in\mathbb{Z}^+$

7) $\lim\limits_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$ $n\in\mathbb{Z}^+$

8) $\lim\limits_{x\to a} c = c$

Ex 1: Evaluate $\lim\limits_{h\to 0}\dfrac{(3+h)^2-9}{h}$.

Ans $= 6$

Ex 2: Given $\lim\limits_{x\to 2} f(x) = 4$ and $\lim\limits_{x\to 2} g(x) = -2$ find $\lim\limits_{x\to 2}(f(x)+5g(x))$.

Ans $= 4+5(-2) = -6$

Ex 3: Find $\lim\limits_{x\to 4}(1+\sqrt{x})(2-6x^2+x^3)$.

$= (1+2)(2-96+74) = (3)(-20) = -60$

Squeeze Thm If $f(x)\leq g(x)\leq h(x)$ when x is near a (except possibly at $x=a$) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

Ex: Use Squeeze Thm to find $\lim_{x \to \infty} \frac{\sin x}{x}$.

$g(x) = \frac{\sin x}{x}$. "The trick is finding f(x) and h(x)."

$-1 \le \sin x \le 1$

$\Rightarrow -\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

$$\underset{\substack{\downarrow \\ 0}}{\lim_{x \to \infty}(-\tfrac{1}{x})} \le \lim_{x \to \infty} \frac{\sin x}{x} \le \underset{\substack{\downarrow \\ 0}}{\lim_{x \to \infty} \frac{1}{x}}$$

$\Rightarrow \lim_{x \to \infty} \frac{\sin x}{x} = 0$

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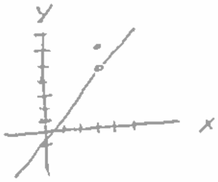
§2.4 The Precise Defn of a Limit

6/27

Consider

$$f(x) = \begin{cases} 2x-1 & x \neq 3 \\ 6 & x = 3 \end{cases}$$



From last time we know " $\lim_{x \to 3} f(x) = ?$ (Ans 5)

"Now I wanna know how close x has to be to 3 so that f(x) differs from 5 by no more than 0.1?"

We want $|f(x) - 5| < 0.1$ so $|x-3| < ?$

Note $x \neq 3$ limits so $|x-3| > 0$.

Q: Can we find δ st $|f(x) - 5| < 0.1$ if $0 < |x-3| < \delta$.

What do we know? f(x)

$$|f(x) - 5| < 0.1$$
$$|(2x-1) - 5| < 0.1$$
$$|2x-6| < 0.1$$
$$2|x-3| < 0.1$$
$$|x-3| < 0.05 = \delta$$

Ans: Yes, $\delta = 0.05$

Q: Can we find δ st $|f(x) - 5| < \overset{\text{arbitrary small \#}}{\varepsilon}$ if $0 < |x-3| < \delta$?

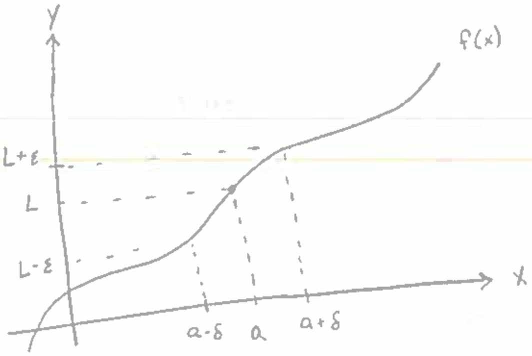
Ans: Yes $\delta = \frac{\varepsilon}{2}$

Defn: Let f be a func defined on some open interval around a (not necessarilly at a). Then we say that

$$\lim_{x\to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a $\delta > 0$ st if

$$0 < |x-a| < \delta \text{ then } |f(x) - L| < \varepsilon.$$



Ex: Prove $\lim_{x\to 1} \frac{2+4x}{3} = 2$.

↑

Do not plug in

Need to find δ.

$$\left|\frac{2+4x}{3} - 2\right| < \varepsilon$$

$$|2+4x-6| < 3\varepsilon$$

$$|4x-4| < 3\varepsilon$$

$$4|x-1| < 3\varepsilon$$

$$|x-1| < \frac{3\varepsilon}{4} = \delta$$

Ex: If $\lim f = L$ and $\lim g = M$, Prove $\lim (f+g) = L+M$.

$$|f+g-(L+M)| \le |f(x)-L| + |g(x)-M| \underset{\substack{\uparrow \\ \because \text{assumptions}}}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$\Rightarrow \delta = \min\{\delta_1, \delta_2\}$ ▫

Defn: Let f be a func defined on some open interval that contains the number a (except possibly a itself). Then

$$\lim_{x \to a} f(x) = \infty \qquad (-\infty)$$

means that for every positive number M there is a positive number δ st if $0 < |x-a| < \delta$ then $f(x) > M$.
$(< -)$

" You guys try to make a defn for $-\infty$ "

Ex: Prove $\lim_{x \to 2} \frac{1}{x^2} = \infty$
↑
N+f δ

$$x^2 - 4 > M \Rightarrow x^2 < \frac{1}{M} \Rightarrow x^2 < \frac{1}{M} \Rightarrow |x| < \frac{1}{\sqrt{M}} = \delta$$

§2.5 Continuity

Defn: A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a).$$

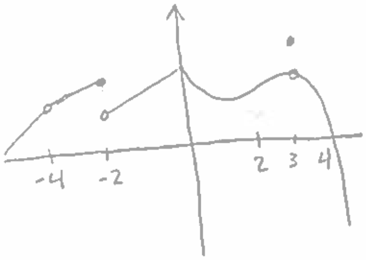
(You can draw the func w/o picking up your pencil)

This implies 3 things:

1) $f(a)$ is defined
2) $\lim_{x \to a} f(x)$ exists
3) $\lim_{x \to a} f(x) = f(a)$

Defn: If f is not continuous we say f is discontinuous.

Ex 1: At which numbers is the graph continuous? Why?



$x = -4$ ∵ $f(-4)$ DNE

$x = -2$ ∵ $\lim_{x \to -2} f(x)$ DNE

$x = 3$ ∵ $\lim_{x \to -2} f(x) \neq f(-2)$

Ex 2: Find pts of discontinuity and explain why.

a) $f(x) = \frac{1}{x+2}$ $x = -2$ ∵ $f(-2)$ DNE

b) $f(x) = \begin{cases} \frac{1}{x+2} & x \neq -2 \\ 1 & x = -2 \end{cases}$ $x = -2$ ∵ $f(-2) \neq \lim$

c) $f(x) = \frac{x+2}{x^2+x-2}$ $x = -2, 1$ ∵ $f(-2)$ and $f(1)$ DNE

d) $f(x) = \begin{cases} \frac{x+2}{x^2+x-2} & x \neq -2, 1 \\ -1/3 & x = -2, 1 \end{cases}$ $x = 1$ ∵ $f(1)$ DNE

Defn A function is cont from the right at a number a
if

$$\lim_{x \to a^+} f(x) = f(a)$$

and cont from the left if

$$\lim_{x \to a^-} f(x) = f(a)$$

Defn: A function f is continuous on an interval if it is cont at every number in the interval. (If f is defined at an end pt we understand it is left/right cont.

#15

Ex:

Ex: use the definition of continuity and properties of limits to show $f(x) = x + \sqrt{x-4}$ is cont on ~~a given interval~~. $[4,\infty)$

Pick $a \in [4,\infty)$

$$\lim_{x\to a}(x+\sqrt{x-4}) = \lim_{x\to a} x + \sqrt{\lim_{x\to a}(x-4)}$$

$$= a + \sqrt{a-4}$$

$$= f(a)$$ for every a in $[4,\infty)$

$\therefore$ cont on $[4,\infty)$

Types of discontinuities: (Assume $x=a$ is a discont of $f(x)$)

1) Jump

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$ and limits are finite

2) Removable

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$ (limit exists)

3) Infinite

$$\lim_{x\to a^-} f(x) = \pm\infty$$

$$\lim_{x\to a^+} f(x) = \pm\infty$$

Ex: Find points of discontinuities and give what type

a) $f(x) = \dfrac{2x}{x-3}$ $x = 3$ infinite

b) $f(x) = \begin{cases} x+1 & x < -1 \\ x^2 & x \geq -1 \end{cases}$ $x = -1$ jump

c) $f(x) = \dfrac{x-1}{x^2-1}$ $x = 1$ removable

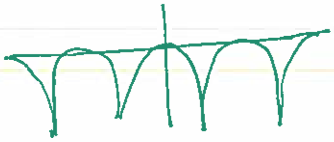
d) $\tan(x)$

$$\tan(x) = \frac{\sin x}{\cos x} \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$x = \frac{(2n+1)\pi}{2}$$ infinite

e) $\ln|\cos x|$ (Not a good exam prob)

$x = n\pi$ infinite



§ 2.6 Limits At Infinity

Ex: $\lim_{x \to \infty} \frac{x^3+3}{x^3-1}$ Recall rules for $\overset{\text{Horz}}{\wedge}$ asymptotes. Same thing!

why? ∵ the numerator ∞ and denom ∞ are both growing at the "same" rate, so $\lim_{x\to\infty} = 1$.

(Once we get into derivatives I can prove this more rigorously)

Defn: The line $y=L$ is called a horz asymp of the curve $y=f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to -\infty} f(x) - L$$

Ex: $\lim_{t\to\infty} \frac{\sqrt{t} + t^2}{2t - t^3} = 0$

Ex: $\lim_{x\to\infty} \frac{\sqrt{x+3x^2}}{4x-1} = \frac{\sqrt{3}}{4}$

Ex: $\lim_{x\to\infty} \sqrt{x} = \infty$

Ex: $\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right) = \lim_{x\to\infty} \frac{x^2+\cancel{x}-x^2}{\sqrt{x^2+1}+x} = \frac{1}{2}$

2.7 Derivatives + Rates of Change

6/29

Defn: The tangent line to the curve $y=f(x)$ at a point $(a, f(a))$ is a line passing through that point w/ slope

$$m = \lim_{h\to 0} \underbrace{\frac{f(a+h)-f(a)}{h}}_{\text{Talk about slope formula first}}$$ (assuming this limit exists).

Ex: Find eqn of tangent line for $f(x) = 4x - x^2$ at the pt $(1,3)$.

$a=1$

$f(1+h) = 4(1+h) - (1+h)^2$

$f(1) = 4-1 = 3$

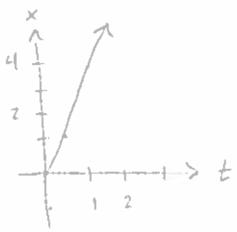
$$\lim_{h\to 0} \frac{4(1+h)-(1+h)^2-3}{h} = \lim_{h\to 0} \frac{4h-2h-h^2}{h} = 2$$

Consider a graph position vs time for a yo-yo

(looks like sine, doesn't need to)

whats the slope here? 0 coincidense

Consider walking in a straight line at 4 ft/s



Intuitively we used his speed for the slope.

(how our position is changing)

If we want to know how fast something is going at a certain time we look at the slope. b/c slope = rate of change

Ex: A partical has a equation of motion $s(t) = 10 + \frac{45}{t+1}$

Find the velocity when $t=4$.

$$s(4+h) = 10 + \frac{45}{4+h+1} \qquad -\frac{45}{(t+1)^2}$$

$$s(4) = 10 + \frac{45}{4+1} = 10+9=19$$

$$\lim_{h\to0} \frac{10+\frac{45}{h+5}-19}{h} = \lim_{h\to0} \frac{45}{h(h+5)} - \frac{9}{h} = \lim \frac{-9h}{h(h+5)} = -\frac{9}{5}$$

Defn: The derivative of a function f at a point a is denoted $f'(a)$ and defined as

$$f'(a) = \lim_{h\to0} \frac{f(a+h)-f(a)}{h}$$

assuming the lim exists.

Ex: Find the derivative of $f(x) = \frac{4}{\sqrt{1-x}}$ at $x=0$

$$\lim_{h\to0} \frac{\frac{4}{\sqrt{1-h}} - \frac{4}{\sqrt{1}}}{h} = \lim_{h\to0} \frac{4-4\sqrt{1-h}}{h\sqrt{1-h}} = \lim_{h\to0} \frac{4(1-(1-h))}{h\sqrt{1-h}(1+\sqrt{1-h})}$$

$$= \lim_{h\to0} \frac{4}{\sqrt{1-h}+1-h} = \frac{4}{2} = 2$$

2.8 Derivative as a function

what if I wanted to find the slope at every point on a curve?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(Definition of a deriv)}$$

Ex: If $f(x) = \sqrt{9-x}$, find a formula for $f'(x)$ & state its domain.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{9-(x+h)} - \sqrt{9-x}}{h} = \lim_{h \to 0} \frac{9-x-h-9+x}{h(\sqrt{9-x-h} + \sqrt{9-x})} = \frac{1}{2\sqrt{9-x}}$$

$$\sqrt{\quad} = [9, \infty)$$

$$\frac{\quad}{0} = \boxed{(9, \infty)} = D$$

Ex: If $f(x) = \frac{x^2-1}{2x-3}$, find $f'(x)$.

Notation: $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x)$

Defn: A func is dif'ble at a if $f'(a)$ exists. It dif'ble on an open interval if it's cont at every number in the interval.

Ex: Where is $f(x) = |x|$ dif'ble?

$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

but $f'(0) = $ DNE.

Thm If f is dif'ble at a, then f is cont at a.

Proof: $f(x)-f(a) = \frac{f(x)-f(a)}{(x-a)}(x-a)$

$$\lim_{x \to a}(f(x)-f(a)) = \lim \frac{f(x)-f(a)}{x-a}(x-a)$$

$$= f'(a) \cdot \lim(x-a)$$

$$= 0$$

$$\lim_{x \to a} f(x) = \lim(f(x)+f(a)-f(a))$$

$$= \lim f(c) + 0$$

$$= f(a)$$

Higher derivatives

Ex: $f(x) = \sqrt{9-x}$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h)-f'(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{2\sqrt{9-x-h}} - \frac{1}{2\sqrt{9-x}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{9-x} - \sqrt{9-x-h}}{2h\sqrt{9-x}\sqrt{9-x-h}} = \lim_{x \to 0} \frac{\cancel{9}-\cancel{x}-\cancel{9}+\cancel{x}+\cancel{h}}{2\cancel{h}\sqrt{9-x}\sqrt{9-x-h}(\sqrt{9-x}+\sqrt{9-x-h})}$$

$$= \frac{1}{2(9-x)(2\sqrt{9-x})} = \frac{1}{4(9-x)^{3/2}}$$

Note:

$s(t) =$ position

$s'(t) =$ velocity

$s''(t) =$ acceleration

6/30

§3.1 Derivatives of poly's and Exponentials

Ex: $f(x) = c$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c-c}{h} = 0$$

Const Rule: The derivative of a constant is zero.

Proof: Ex 1.

Power Rule Let $n \in \mathbb{R}$ then $\frac{d}{dx}(x^n) = n x^{n-1}$

Ex 2: $f(x) = \frac{1}{\sqrt[5]{x^3}}$

Const Mult Rule: If c is a const and f is dif'ble then $\frac{d}{dx}(c f(x)) = c \frac{d}{dx} f(x)$.

Ex: $f(x) = 4x^{3/2}$

Sum Rule: If f and g are both dif'ble then

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

Ex: $f(x) = 3x^4 + 5x^3$

Diff Rule: If f and g are both dif'ble then

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x).$$

Ex: $f(x) = 3x^4 - 5x^3$

Defn: e is the number st $\lim_{h \to 0} \frac{e^h - 1}{h}$.

Deriv of Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Ex: $f(x) = e^x + x^4 - x^3$

Ex: Find f' and f''.

a) $f(x) = x^{2.4} + e^{2.4}$

b) $f(x) = \dfrac{\sqrt[3]{x} - 2xe^{x}}{x}$

c) $f(x) = e^{x+1} + 1$

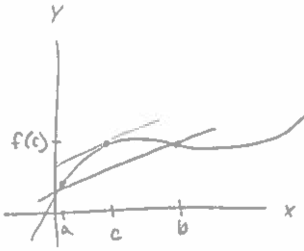
7/03

§4.2 Mean Value Thm

Mean Value Thm: Let f be function that is:

1) cont on $[a,b]$
2) dif'ble on (a,b)

Then there is $c \in (a,b)$ $(a<c<b)$ st $f'(c) = \frac{f(b)-f(a)}{b-a}$.



Rolles Thm: Let f be a func that is:

1) cont on $[a,b]$
2) dif'ble on (a,b)
3) $f(a) = f(b)$

Then there is $c \in (a,b)$ st $f'(c) = 0$.

Ex: Verify that $f(x) = 2x^2 - 3x + 1$ satisfies conditions of MVT and find all #'s c that satisfy the conclusion.

$$f'(x) = 6x - 3$$

$$f'(c) = \frac{f(2) - f(0)}{2-0}$$

$$6c - 3 = \frac{8-6+1-1}{2} = 1$$

$$6c = 4$$

$$c = 2/3$$

Ex: Verify $f(x) = x^3 - 2x^2 - 4x + 2$ satisfies conditions of Rolle's Thm on $[-1, 3]$ and find all #'s c that satisfy the conclusion.

$$f'(x) = 3x^2 - 4x - 4$$

$$f'(c) = 0$$

$$3c^2 - 4c - 4 = 0$$

$$c = \frac{4 \pm \sqrt{16 + 16 \cdot 3}}{6} = \frac{4 \pm 4 \cdot 2}{6} = 2, -2/3$$

§3.2. Product and Quotient Rule

7/03

$$\frac{d}{dx}[f(x)g(x)] \neq f'(x)\,g'(x)$$

Ex: $f(x) = x \qquad f'(x) = 1$

$g(x) = 1 \qquad g'(x) = 0$

$f(x)\,g(x) = x \qquad f'(x) \cdot g'(x) = 0$

$$\frac{d}{dx}(f(x)g(x)) = 1$$

$$(f \cdot g)' = \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)g(x)}{h} = \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - g(x+h)f(x) + g(x+h)f(x) + fg}{h}$$

$$= \lim_{h \to 0} \frac{g(x+h)[f(x+h) - f(x)] + f(x)[g(x+h) - g(x)]}{h}$$

$$= \lim_{h \to 0} \left[g(x+h)\left(\frac{f(x+h)-f(x)}{h}\right) + f(x) \cdot \left(\frac{g(x+h) - g(x)}{h}\right)\right]$$

$$= g(x)\,f'(x) + f(x)g'(x)$$

The Product Rule: If f and g are dif'ble, then

$$\frac{d}{dx}(f(x)\,g(x)) = f'(x)\,g(x) + f(x)\,g'(x)$$

Ex: $J(v) = (v^3 - 2v)(v^{-4} + v^{-2})$

$f(v) = v^3 - 2v \qquad f'(v) = 3v^2 - 2$

$g(v) = v^{-4} + v^{-2} \qquad g'(v) = -4v^{-5} - 2v^{-3}$

$$J'(v) = (v^3 - 2v)(-4v^{-5} - 2v^{-3}) + (v^{-4} + v^{-2})(3v^2 - 2)$$

If you can guess $\frac{d}{dx}\frac{f(x)}{g(x)} \neq \frac{f'(x)}{g'(x)}$.

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{f(x+h)g(x) - g(x+h)f(x)}{h(g(x+h)g(x))}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - g(x+h)f(x)}{h\, g(x+h)\, g(x)}$$

$$= \lim_{h\to c} \left[\left(g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}\right)\frac{1}{g(x+h)g(x)}\right]$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Quotient Rule: If f and g are both dif'ble then:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Ex: $F(x) = \frac{x^4 - 5x^3 + \sqrt{x}}{x^2}$

$f(x) = x^4 - 5x^3 + \sqrt{x}$ $\quad f'(x) = 4x - 15x^2 + \frac{1}{2\sqrt{x}}$

$g(x) = x^2$ $\quad g'(x) = 2x$

$$F'(x) = \frac{(4x - 15x^2 + \frac{1}{2\sqrt{x}})x^2 - (x^4 - 5x^3 + \sqrt{x})(2x)}{x^4}$$

or rewrite $F(x)$ and use power rule.
(see if two methods match)

7/05

§ 3.3 Derivatives of Trig Funcs

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

Useful: $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ $\quad \lim\limits_{h \to 0} \frac{\cos\theta - 1}{\theta} = 0$

$$\frac{d}{dx} \cos x = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \cos x \frac{\cos h - 1}{h} - \lim_{h \to 0} \sin x \frac{\sin h}{h} = -\sin x$$

Ex: Prove $\frac{d}{dx} \tan x = \sec^2 x$ using quotient rule.

Ex: Find the 23rd deriv of $f(x) = \sin x$.

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x = f^{(23)}(x)$$

Ex: Find $\lim\limits_{x \to 0} \frac{3x}{\sin 6x} = \lim\limits_{x \to 0} \frac{1}{2 \sin 6x / 6x} = \frac{1}{2} \frac{1}{\lim\limits_{x \to 0} \frac{\sin 6x}{6x}} = \frac{1}{2}$

Ex: $\lim\limits_{x \to 0} \csc x \sin(\sin x) = 1$

Ex: $\lim\limits_{\theta \to 0} \frac{\cos\theta - 1}{\sin\theta} = \lim\limits_{\theta \to 0} \frac{(\cos\theta - 1)/\theta}{\sin\theta/\theta} = \frac{0}{1} = 0$

Ex: Differentiate $y = \frac{\cos x}{1-\sin x}$ twice.

$$y' = \frac{-\sin x(1-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$$

$$= \frac{1}{1-\sin x}$$

$$y'' = \frac{0(1-\sin x) - 1(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{\cos x}{(1-\sin x)^2}$$

§ 3.4 Chain Rule

Chain Rule: $\frac{d}{dx}\left(f(g(x))\right) = \underset{\uparrow \text{ not a deriv}}{f'(g(x))}\, g'(x)$ or $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Ex: Differentiate

a) $f(x) = \frac{1}{1-\sin x}$

b) $\csc x$

c) $\sec x$

d) $\cot x$

e) $e^{\sqrt{x}}$

f) $(2x^3+5)^4$

g) $(t+1)^{2/3}(2t-1)^3$

Ex: Find y' and y'' for $y = e^{e^x}$

$y' = e^{e^x} e^x$

$y'' = e^{e^x} e^{2x} + e^{e^x} e^x$

Derivative of an exponential

$$\frac{d}{dx}\left(b^x\right) = b^x \ln b$$

where $b > 0$.

Pf: $\frac{d}{dx}\left(b^x\right) = \frac{d}{dx} e^{(\ln b)x}$

$= e^{(\ln b)x} \cdot \ln b$

$= b^x \cdot \ln b$

§3.5 Implicit Differentiation

7/06

What if we want to take deriv of $x^2+y^2=4$.

Can't solve for y.

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}4$$

$$2x+2y\frac{dy}{dx}=0$$

$$2y\frac{dy}{dx}=-2x$$

$$y'=-x/y$$

Ex: Find y' if $x^3-xy^2+y^3=1$

Ex: Find slope of the tangent line for $y\sin x=x\cos 2y$ at $(\frac{\pi}{2},\pi)$

$$y'\sin x+y\cos x=\cos 2y-x\sin 2y\cdot 2y'$$

$$y'=\frac{\cos 2y-y\cos x}{\sin x+x\sin 2y}$$

$$y'\Big|_{\substack{x=\pi/2\\ y=\pi}}=\frac{\cos 2\pi}{\sin \pi/2}=1$$

Ex: Find y'' if $\sin y+\cos x=1$.

$$y'\cos y-\sin x=0$$

$$y'=\frac{\sin x}{\cos y}$$

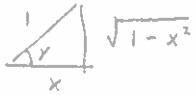
$$y''=\frac{-\sin^2 x-\cos^2 y}{\cos^2 y}$$

Derivatives of Inv Trig Funcs

$$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1}x)=\frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$$

If $y = \cos^{-1} x \Rightarrow \cos y = x \Rightarrow$ 

$\Rightarrow -y' \sin y = 1$

$\Rightarrow y' = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}/1} = -\frac{1}{\sqrt{1-x^2}}$

§3.6 Derivs of Logarithmic Funcs

$$\frac{d}{dx} \log_b x = \frac{1}{x \ln b}$$

Proof: $y = \log_b x$

$$b^y = b^{\log_b x} = x$$

$$\frac{d}{dx} b^y = \frac{d}{dx} x$$

$$b^y \ln b \cdot y' = 1$$

$$y' = \frac{1}{b^y \ln b}$$

Ex: $\frac{d}{dx} \ln x = ?$ (Memorize!)

Ex: $f(x) = \ln \frac{(2x+1)^5}{\sqrt{x^2+1}} = 5 \ln(2x+1) - \frac{1}{2} \ln(x^2+1)$

$$f'(x) = \frac{5}{2x+1} \cdot 2 - \frac{1}{\cancel{2}} \frac{1}{x^2+1} \cdot \cancel{2}x = \frac{10}{2x+1} - \frac{x}{x^2+1}$$

Steps of log dif

1) Take ln of both sides
2) Use log rules to simplify
3) Differentiate implicitly wrt x
4) solve resulting eqn for y'.

Ex: Differentiate $y = x^{\sqrt{x}}$

$$\ln y = \sqrt{x} \ln x$$

$$\frac{1}{y} y' = \frac{1}{2\sqrt{x}} \ln x + \frac{\sqrt{x}}{x} = \frac{1}{2\sqrt{x}} \ln x + \frac{1}{\sqrt{x}}$$

$$y' = x^{\sqrt{x}} \left(\frac{1}{2\sqrt{x}} \ln x + \frac{1}{\sqrt{x}} \right)$$

Ex: $y = \ln x^{\cos x}$

$\ln y = \ln(\ln x) \cdot \cos x$

$\frac{y'}{y} = \frac{1}{\ln x} \cdot \frac{1}{x} \cos x - \ln(\ln x) \sin x$

$y' = \ln x^{\cos x} \left(\frac{\cos x}{x \ln x} - \sin x \ln(\ln x) \right)$

Ex: Prove Power Rule using logrithmic diff

$y = x^n$

$\ln y = n \ln x$

$\frac{1}{y} y' = n \frac{1}{x}$

$y' = n \frac{y}{x} = n \frac{x^n}{x} = n x^{n-1}$

7/07

§4.4 l'Hospital'Rule

Sps we want to find $\lim_{x\to\infty} \frac{\ln x}{x-1}$ or $\lim_{x\to 1} \frac{\ln x}{x-1}$.

These are indeterminite forms: $\frac{\infty}{\infty}$ or $\frac{0}{0}$ or $\infty - \infty$, 0^∞, ∞^0, 1^∞

<u>L'Hospital's Rule</u> Sps f and g are dif'ble and $g'(x) \neq 0$ on an open interval that has the pt a in it. Sps that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

OR

$$\lim_{x\to a} f(x) = \infty \quad \text{and} \quad \lim_{x\to a} g(x) = \infty$$

then $\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.}$

Ex: $\lim_{x\to\infty} \frac{\ln x}{x-1} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

Ex: $\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1$

Ex: $\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^4} = \lim_{x\to 0} \frac{-\sin x + x}{4x^3} = \lim_{x\to 0} \frac{-\cos x + 1}{12x^2} = \lim_{x\to 0} \frac{\sin x}{24x} = \lim_{x\to} \frac{\cos x}{24} = \frac{1}{24}$

Ex: $\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \lim_{x\to\infty} \frac{3x^2}{e^{x^2}(2x)} = \lim_{x\to\infty} \frac{3x}{2e^{x^2}} = \lim_{x\to\infty} \frac{3}{4xe^{x^2}} = 0$

Ex: $\lim_{x\to 0} (\csc x - \cot x) = \lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) = \lim_{x\to 0} \left(\frac{1 - \cos x}{\sin x} \right) = \lim_{x\to 0} \frac{\sin x}{\cos x} =$

Ex: $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right) = \lim_{x\to 0^+} \left(\frac{e^x - 1 - x}{x(e^x - 1)} \right) = \lim_{x\to 0^+} \frac{e^x - 1}{e^x + xe^x - 1} = \lim_{x\to 0^+} \frac{e^x}{2e^x + xe^x}$

$= \lim_{x\to 0^+} \frac{1}{2+x} = 1/2$

$\Rightarrow \lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right) = e^{1/2}$

Indet forms: ∞^0, 0^∞, 1^∞ => set = to y, take ln
find lim
e^{result}

Ex: $\lim\limits_{x \to 0^+} x^{\sqrt{x}}$

↑ why is + important?

$y = x^{\sqrt{x}} \Rightarrow \ln y = \sqrt{x} \ln x$

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \sqrt{x} \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/\sqrt{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}}$$

$$= \lim_{x \to 0^+} - \frac{2x^{3/2}}{x} = \lim_{x \to 0^+} (-2x^{1/2}) = 0$$

$$\Rightarrow \lim_{x \to 0^+} x^{\sqrt{x}} = e^0 = 1$$

Ex: $\lim\limits_{x \to \infty} \left(1 + \frac{a}{x}\right)^{bx}$ a = const b = const

$y = \left(1 + \frac{a}{x}\right)^{bx} \Rightarrow \ln y = bx \ln\left(1 + \frac{a}{x}\right)$

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln(1 + a/x)}{1/bx} = \lim_{x \to \infty} \frac{\frac{1}{1+\frac{a}{x}}\left(-\frac{a}{\cancel{x^2}}\right)}{-\frac{1}{\cancel{bx^2}}} \begin{matrix} \cdot(-b\cancel{x^2}) \\ \cdot(-bx^2) \end{matrix}$$

$$= \lim_{x \to \infty} \frac{1}{1+\frac{a}{x}}(ab) = \lim_{x \to \infty} \frac{ab}{1 + a/x} = ab$$

$$\Rightarrow \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{bx} = e^{ab}$$

§3.9 Related Rates

7/12

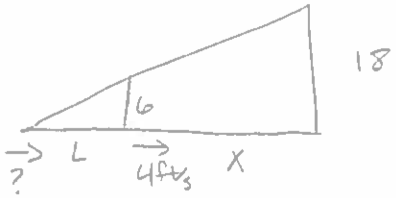
Solving Relate Rates

1) Draw it
2) Identify geometric shape (label)
3) Identify varying parameters and corresponding variables
4) Identify useful formula for shape and parameters
5) Get rid of extraneous variables
6) Differentiate wrt time
7) sub values in and solve

Ex: A pebble is dropped into a calm pond causing circular ripples to form. The radius r is increasing at a rate 1 ft/s. When the radius is 4ft, at what rate is the area of disturbed water changing?

$\frac{d}{dt}(A = \pi r^2) \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(4)(1) = 8\pi$

Ex: A 6ft man walkss toward a 18ft lamp post at a rate 4ft/s. When he is 10 ft away from the lampost, how fast is the tip of his shadow moving?



$\frac{18}{X+L} = \frac{6}{L} \Rightarrow 18L = 6X + 6L$

$12L = 6X$

$12 \frac{dL}{dt} = 6 \frac{dX}{dt}$

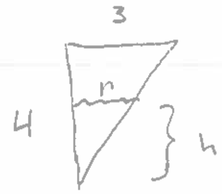
$\frac{dL}{dt} = \frac{1}{2}(4) = -2 \text{ ft/s}$

Ex: A water tank is shaped like an inverted cone w/ base radius 2m and hieght 4m. If water is being pumped into the tank at a rate $2 \, m^3/min$, find the rate at which the water level is rising when the water is 3m deep.

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{3}{4}h\right)^2 h$$

$$V = \frac{3}{16}\pi h^3$$



$$\frac{3}{r} = \frac{4}{h}$$

$$r = \frac{3}{4}h$$

$$\frac{dV}{dt} = \frac{3}{16}\pi(3h^2)\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{9}{16}\pi h^2 \frac{dh}{dt}$$

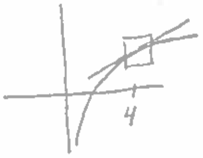
$$2 = \frac{9}{16}\pi(3)^2\frac{dh}{dt} \Rightarrow \boxed{\frac{dh}{dt} = \frac{32}{81\pi}}$$

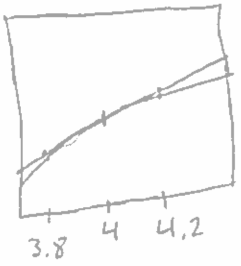
Are these useful? Yes. But you'll never see stuff like this outside Calc. What you will see is: $PV = NRT$. What happens to the temperature if I decr the pressure?

7/13

§ 3.10 Linear Approx's and Differentials

Linear Approx





Recall: finding the eqn of a tangent line.
The pt on the tangent line is close to the one we want and easier to find.

Let $L(x)$ = eqn of tangent line

$$(y - y_0) = m(x - x_0)$$

$$(L(x) - f(a)) = f'(a)(x - a)$$

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$ Linearization of a function at a pt a

$\Rightarrow f(x) \approx L(x) = f(a) + f'(a)(x-a)$ when x is close to a.

Ex: Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $a=0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$.

$$L(x) = f(0) + f'(0)(x-0) = \sqrt{1} + \frac{1}{2}x = 1 + \frac{x}{2}$$

$$f'(x) = \frac{1}{2\sqrt{1-x}}$$

$$\sqrt{.9} = f(-0.1) \approx 1 - \frac{0.1}{2} = 1 - 0.05 = 0.95$$

$$\sqrt{.99} = f(-0.11) \approx 1 - \frac{0.11}{2} = 1 - 0.055 = 0.945$$

Ex: Find the linear approximation of $\sin(x)$ when $a=0$.

$$L(x) = \sin(0) + \cos(0)(x+0) = x$$

$\Rightarrow \sin x \approx x$ around $a=0$

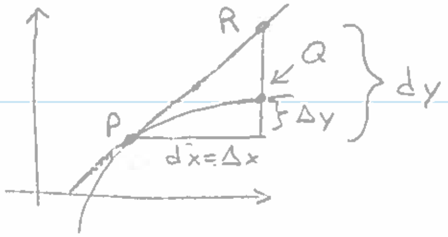
Differentials - Great error analysis

$$\frac{dy}{dx} = f'(x)$$

$$dy = f'(x)\,dx$$ ← differential

change in y

change in x



Ex: a) Compute Δy and dy when $y = x^2 - 4x$ for x changing from $x=3$ to $x=3.5$

$y(3) = 3^2 - 4(3) = -3$

$y(3.5) = 3.5^2 - 4(3.5) = -1.75$

$\Delta y = -3 - (-1.75) = -1.25$

$dy = (2(3)-4)(3.5-3) = 2(.5) = 1$

b) What about if x changes from $x=3$ to $x=3.2$?

$y(3) = -3$

$y(3.2) = -2.56$

$\Delta y = 0.44$

$dy = (2(3)-4)(3.2-3) = 0.4$

Ex: Write the differential form of $y(x) = \sqrt{1 + \ln x}$.

$$\frac{dy}{dx} = \frac{1}{2}(1+\ln x)^{1/2} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{1+\ln x}}$$

$$dy = \frac{1}{2x\sqrt{1+\ln x}}\,dx$$ (Note: not linear!)

Ex: Use linearization and differentials to find $\sqrt{2}$.

Use $f(x) = \sqrt{x}$ with the pt $(1,1)$.

Linearization	vs	Differential
	$f'(x) = \frac{1}{2\sqrt{x}}$	
$L(x) = f(1) + f'(1)(x-1)$ $= 1 + \frac{1}{2}(x-1)$		$dy = \frac{1}{2\sqrt{x}}\,dx$
$L(2) = 1 + \frac{1}{2}(2-1) = 1.5$		Going from $x=1$ to $x=2$ $dy = \frac{1}{2\sqrt{1}} \cdot (2-1) = .5$
		$\sqrt{2} \approx \sqrt{1} + dy = 1.5$
• Finds pt close to needed pt on tangent line		• estimates change in y using tangent line slope
• More intuitive for estimates		• Good for error analysis

Ex: The edge of a cube is found to be 30 cm w/ a possible measurement error of 0.1 cm. Use differentials to estimate the a) maximum possible error, b) relative error and c) percentage error for the volume of a cube

a) $V = s^3$

$$\frac{dV}{ds} = 3s^2$$

$$dV = 3s^2\,ds$$

$$dV = 3(30)^2(0.1) = (2700)(0.1) = 270\text{ cm}^3$$

$s = 30 \pm 0.1$

$V = 27000 \pm 270$

b) Divide error by total volume

$$\frac{dV}{V} = \frac{270}{27000} = 0.01$$

c) 1%

Steps: Let $x = x_0$ be the hard pt and $x = a$ be the easy one.

Linearization

Differentials

1) define $f(x)$

2) find $f'(x)$

Linearization:

3) $L(x) = f(a) + f'(a)(x-a)$

4) $f(x_0) \approx L(x_0)$

Differentials:

3) $dy = f'(x)\,dx$

4) $dy = f'(a)(x_0 - a)$

5) $f'(x_0) \approx f(a) + dy$

Ex: a) $(2.001)^5$

b) $e^{-0.015}$

c) $\ln(1.05)$

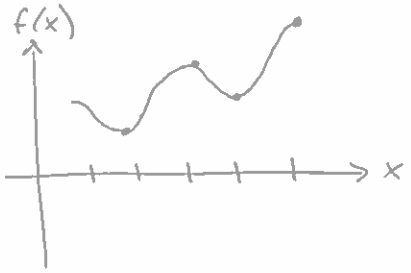
§4.1 Max and Min Values

7/14

Defn: Let be a in the domain of a cont func f. Then f(c):

- abs max if $f(c) \geq f(x)$ $\forall x$ in the domain of f
- abs min if $f(c) \leq f(x)$ " "
- local max if $f(c) \geq f(x)$ $\forall x$ near c
- local min if $f(c) \leq f(x)$ " "

Ex: sketch the graph of a cont func on [1,5] w/ abs max @ 5, abs min @ 2, local max @ 3 and local min @ 2 and 4.



Note: End pts can be abs mins/maxes but not local.

Defn: Extrema means local/abs min/max's

§4.3 How Derivatives Affect the shape of a Graph + (4.1)

What do derivatives tell us about the shape of a graph?

Incr / Decr Test:

a) If $f'(x) > 0$ on an interval, then f is increasing on that interval

b) If $f'(x) < 0$ " " decr " "

What about when $f'(0) = 0$?

Fermat's Thm If f has a local max or min at c and $f'(c)$ exists, then $f'(c) = 0$.

Defn: A critical pt is a # c st $f'(c) = 0$ or DNE.

Ex: Find the critical numbers of $f(x) = x^{4/5}(x-4)^2$

$$f'(x) = \frac{\frac{4}{5}(x-4)^2 + 2x(x-4)}{x^{1/5}}$$

CP: $x = 0$ min or max?

$x = 4, 8/7$

How can we tell?

First derivative test sps c is a critical # of a cont func f

a) If f' changes from + to - at c, then f(c) = local max

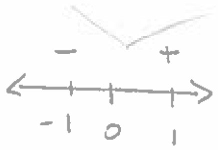
b) If f' changes from - to + at c, then f(c) = local min

c) If f' does not change we have no extrema.

Ex: Find critical pts of $f(x) = x^2$ and label extrema.

$f'(x) = 2x = 0$

$x = 0$



CP: $x = 0$ abs min

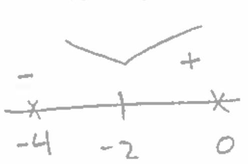
Ex Find critical pts for $f(x)$ and label extrema.

$f(x) = xe^{x/2}$

a) $f'(x) = xe^{x/2} \cdot \frac{1}{2} + e^{x/2} = e^{x/2}\left(\frac{x}{2} + 1\right) = 0$

$x = -2$

CP : $x = -2$



Abs min: $\left(-2, \frac{-2}{e}\right)$

b) $f(x) = \sqrt[3]{x}$ CP : $x = 0$ nothing

c) $f(x) = -|x|$ CP: $x = 0$ abs max: $(0,0)$

d) $f(x) = 2x^3 - 3x^2 - 12x + 1$

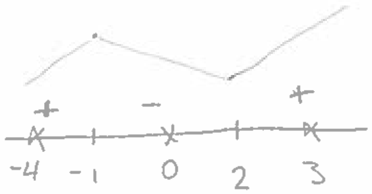
i) on $\mathbb{R}$

$f'(x) = 6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

CP: $x = -1, 2$



$(-1, 8)$ local max

$(2, -19)$ local min

ii) on $[-4, 3]$

$f(-4) = -127$ abs min $(-4, -127)$

$f(-1) = 8$ abs max $(-1, 8)$

$f(2) = -19$ local min $(2, -19)$

$f(3) = -8$ nothing

What does f'' give us?

Concavity Test

☺ a) If $f''(x)>0$ for all x on an interval f is concave up on that interv

☹ b) If $f''(x)<0$ 〃 〃 down 〃 〃

Defn: If $f''(0)=0$ we call c an inflection pt.

The second derivative test sps f'' is cont near $x=c$.

a) If $f'(c)=0$ and $f''(c)>0$ then $f(c)=$ local min

b) If $f''(c)=0$ and $f''(c)<0$ then $f(c)=$ local max.

Ex: Find critical pts, extrema and intervals on which f(x) is concave up or down. Use this to sketch a graph.

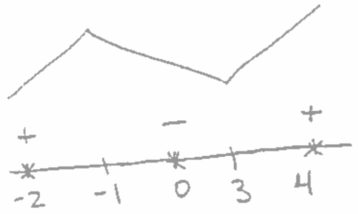
a) $f(x)=x^3-3x^2-9x+4$

$f'(x)=3x^2-6x-9=0$

$x^2-2x-3=0$

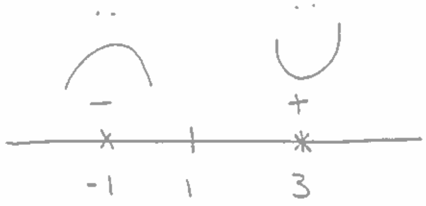
$(x-3)(x+1)=0$

CP: $x=-1,3$



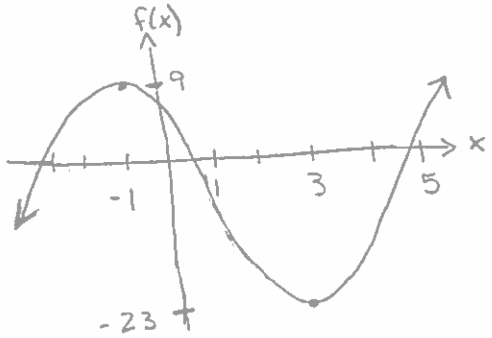
local max: $(-1,9)$

local min: $(3,-23)$



$f''(x)=6x-6=0$

$x=1$



b) $f(x) = x^4 - 2x^2 + 3$

§4.5 Summary of Curve Sketching

4/17

1) Domain
2) Intercepts
   x-int: $y=0$ (Not always viable)
   y-int: $x=0$ (Usually easy)
3) Asymptotes
   horz: $\lim_{x\to\pm\infty} f(x) = L < \infty$
   vert: $\lim_{x\to a^{\pm}} f(x) = \pm\infty$
   } Talk more deeply
4) Find Critical pts
   $f'(x) = 0$ or DNE
5) Find Intervals of Incr/Decr
   real line test
6) Find extrema
7) Find pts of inflextion
   $f''(x) = 0$
8) Intervals of concave up/down
9) sketch

Ex: Graph $f(x) = \frac{x^2+5x}{x^2-25}$

Domain: $x \neq \pm 5$

y-int: $f(x) = \frac{x}{x-5} \Rightarrow f(0) = 0$ $(0,0)$

x-int: $x = 0$ $x = 0$ $(0,0)$

Horz asymp: $\lim_{x\to\infty} f(x) = 1$
$\lim_{x\to-\infty} f(x) = 1$ $y = 1$

Vert asymp: $x = 5$

$f'(x) = 0 \Rightarrow \frac{x-5-x}{(x-5)^2} = -\frac{5}{(x-5)^2} \neq 0$

CP: $x = 5$

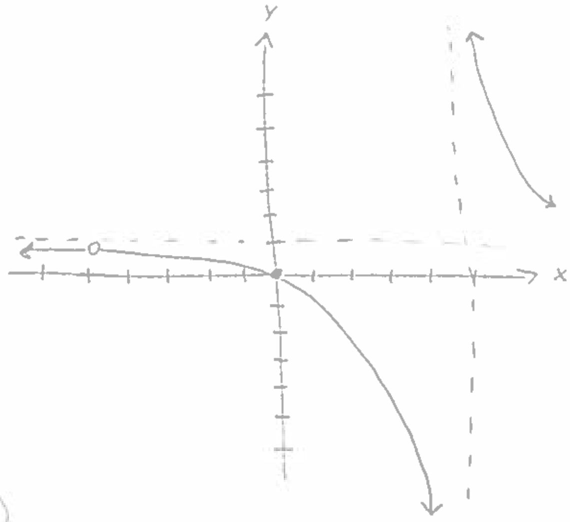
Incr: DNE
Decr: $(-\infty, 5) \cup (5, \infty)$

Extrema: None

$f''(x) = 0 \Rightarrow \frac{10}{(x-5)^3} \neq 0$

Inf pt: $x = 5$

Concave Up: $(5, \infty)$
Concave Down: $(-\infty, 5)$



Ex: $f(x) = \frac{x^2+1}{x^2-x-12}$ Bad example

Ex: $f(x) = \frac{\ln(x)}{\sqrt{x^2+1}}$

Ex: $f(x) = xe^x$

§4.7 Optimization Problems

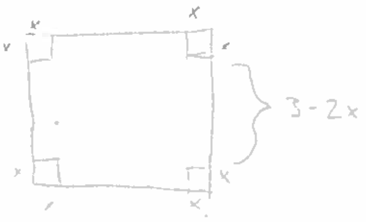
4/18

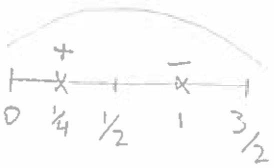
This is where knowing how to find a min/max is handy irl.

Solving Optimization Problems

1) Draw it
2) Introduce notation and label drawing
3) Write a useful equation
4) Eliminate extra variables
5) Find absolute min's and max's (not local)
6) Check if answers are reasonable

Ex: A box with an open top is to be constructed from a sqr piece of cardboard 3 ft wide by cutting a square out of each corner and bending up the sides. Find the largest volume the box can have.





$$V = (3-2x)^2 x = 9x - 12x^2 + 4x^3$$

$$V' = 9 - 24x + 12x^2 = 0$$

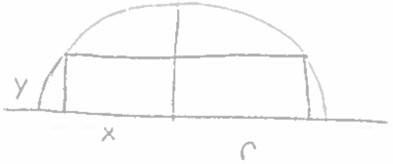
$$4x^2 - 8x + 3 = 0$$

$$x = \frac{8 \pm \sqrt{64 - 4(4)(3)}}{2(4)} = \frac{8 \pm 4}{8} = \frac{1}{2},$$

$x = \frac{1}{2}$ maximizes V

$$V_{max} = (3 - 2 \cdot \tfrac{1}{2})^2 \tfrac{1}{2} = 4 \cdot \tfrac{1}{2} = 2$$

Ex: Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.



$A = (2x)y$ $\qquad x^2+y^2=r^2$

$A = 2x\sqrt{r^2-x^2}$ $\qquad y = \sqrt{r^2-x^2}$

Domain: $0 \le x \le r$

Is r changing? No.

$$A' = 2\sqrt{r^2-x^2} + x(r^2-x^2)^{-1/2}(-2x)$$

$$= \frac{2(r^2-x^2-x^2)}{\sqrt{r^2-x^2}} = \frac{2(r^2-2x^2)}{\sqrt{r^2-x^2}} = 0$$

$$r^2-2x^2=0$$

$$x^2 = \frac{r^2}{2}$$

$$x = r/\sqrt{2} \quad \Rightarrow \quad y = \sqrt{r^2-\left(\frac{r}{\sqrt{2}}\right)^2} = \frac{r}{\sqrt{2}}$$

$$A = 2\left(\frac{r}{\sqrt{2}}\right)\left(\frac{r}{\sqrt{2}}\right) = r^2$$

Ex: Find 2 numbers whose product is 100 and sum is a min (in $\mathbb{R}^+$).

$$\begin{cases} xy=100 \Rightarrow y = \frac{100}{x} \\ x+y=m \end{cases}$$

$$m = x + \frac{100}{x}$$

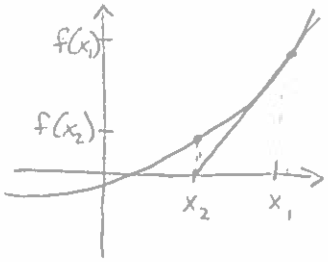
$$m = 1 - \frac{100}{x^2} = 0$$

$$x^2=100 \Rightarrow x=10 \quad \Rightarrow \quad y = \frac{100}{10} = 10$$

7/19

§4.8 Newton's Method

Here is a curve:



Goal: find root

Recall linearization: pt on tangent line is close to pt on curve

Eqn of tangent line: $f(x) - f(x_1) = f'(x_1)(x - x_1)$

$$f(x) = f(x_1) + f'(x_1)(x - x_1)$$

Root: $\overset{f(x_2)=}{0} = f(x_1) + f'(x_1)(x_2 - x_1)$

$\vdots$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Find an eqn for x_3.

What can we say about x_{n+1} = ?

Newton's Method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

If we iterate we get closer, right?

Ex: Find the third approximation x_3 to the root of the eqn $x^3 - 2x - 5 = 0$ (Use $x_1 = 2$).

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

$$x_2 = 2 - \frac{8-4-5}{12-2} = 2 + \frac{1}{10} = 2.1$$

$$x_3 = 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} = 2.0946$$

Ex: Use Newton's Method to find $\sqrt[6]{2}$ correct to 8 decimal places.

$$x^6 - 2 = 0$$

$$x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5}$$

$\sqrt[6]{2} \approx 1.12246205$

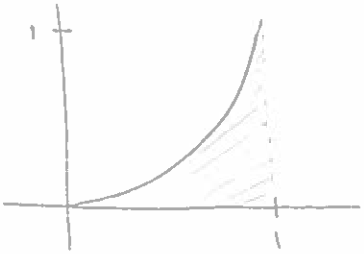
Ex: Find the root of $\cos x = x$ correct up to 6 decimal places.

$x_5 = 0.73908513$

7/20

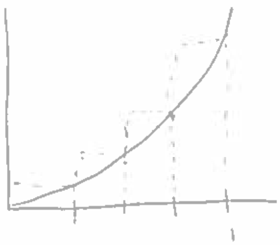
§5.1 Area and Distance

How can we find the area for



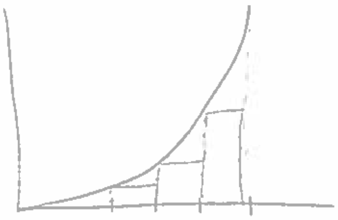
?

We can estimate it using rectangles.



Too much

vs



Too little

Base of □ : ¼

height: $\left(\frac{1}{4}\right)^2, \left(\frac{1}{2}\right)^2, \left(\frac{3}{4}\right)^2, (1)^2$

Base : ¼

height : $(0)^2, \left(\frac{1}{4}\right)^2, \left(\frac{1}{2}\right)^2, \left(\frac{3}{4}\right)^2$

$$R_4 = U_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}(1)^2$$

$$= .46875$$

$L_4 = .21875$ $\Rightarrow$ $0.21875 < A < 0.46875$

If we use more rectangles, we could get a better axpproximation.

Show Figure 8-9 pg 369

If $n=10$

Base : $\frac{1}{10}$

height : $\left(\frac{1}{10}\right)^2, \left(\frac{2}{10}\right)^2, \ldots, (1)^2$

Base : $\frac{1}{10}$

height : $0, \left(\frac{1}{10}\right)^2, \ldots, \left(\frac{9}{10}\right)^2$

$R_{10} = 0.385$

$L_{10} = 0.285$

Can we develop a formula for any # of □'s?

$$U_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \dots + \frac{1}{n}\left(\frac{n}{n}\right)^2 = \frac{1}{n}\left(\frac{1}{n^2}\right)\left(1^2 + 2^2 + \dots + n^2\right)$$

$$= \frac{1}{n^3}\sum_{K=1}^{n} K^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{(n+1)(2n+1)}{6n^2}$$

What does it mean physically if we let n go to ∞? Our base goes to zero.

$$\lim_{n\to\infty} U_n = \frac{(n+1)(2n+1)}{6n^2} = \frac{2}{6} = \frac{1}{3}$$

← doesn't really count

$$L_n = \frac{1}{n^3}\left(0^2 + 1^2 + \dots + (n-1)^2\right) = \frac{1}{n^3}\sum_{K=1}^{n-1} K^2 = \frac{1}{n^3}\,\frac{(n-1)(n)(2n-1)}{6}$$

$$= \frac{(n-1)(2n-1)}{6n^2} = \frac{1}{3}$$

So our over estimate and our underestimate match...

(squeeze Thm)

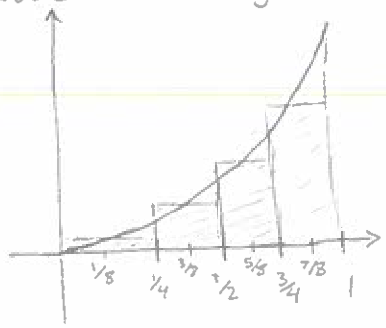
$A = 1/3$

Defn: The area A of a region $\overset{s}{\wedge}$ under a continuous func f is

$$A = \lim_{n\to\infty} L_n \overset{\text{or}}{=} \lim_{n\to\infty} U_n = \lim_{n\to\infty}\left(f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x\right)$$

$$= \lim_{n\to\infty} \Delta x \sum_{K=1}^{n} f(x_K)$$

We used the left side of the □ for height, we also used the right. What else can we use? The Middle.



$$A \approx M_4 = \frac{1}{4}\left(\frac{1}{8}\right)^2 + \frac{1}{4}\left(\frac{3}{8}\right)^2 + \frac{1}{4}\left(\frac{5}{8}\right)^2 + \frac{1}{4}\left(\frac{7}{8}\right)^2 = 0.32815$$

In general:

$$A \approx M_n = \Delta x\, f(x_1^*) + \Delta f(x_2^*) + \dots + \Delta x\, f(x_n^*)$$

$$x_1 < x_1^* < x_2$$

§5.2 Definite Integrals

Defn: A definite integral of f from a to b is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\tilde{x}_i)\,\Delta x = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(\tilde{x}_i)$$

provided that the limit exists and gives the same value for all possible choices of $\tilde{x}_i$. If it does exist, then we say f is integrable on a to b.

Ex: a) Evaluate the Riemann Sum for $n=6$ for $f(x) = x^3 - 6x$ on the interval $[0, 3]$.

$$U_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x = \left(f(\tfrac{1}{2}) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)\right) \underbrace{\Delta x}_{1/2}$$

$$= -3.9375$$

b) Evaluate $\int_a^b (x^3 - 6x)\,dx$.

$$\int_a^b (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} f\left(\frac{3k}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left[\left(\frac{3k}{n}\right)^3 - 6\left(\frac{3k}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left[\frac{27}{n^3} k^3 - \frac{18}{n} k\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \left[\frac{27}{n^3} \sum_{k=1}^{n} k^3 - \frac{18}{n} \sum_{k=1}^{n} k\right] \qquad \text{(see pg 381)}$$

$$= \lim_{n\to\infty} \frac{3}{n} \left(\frac{27}{n^3} \left(\frac{n(n+1)}{2}\right)^2 - \frac{18}{n} \left(\frac{n(n+1)}{2}\right)\right)$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2} \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{4} \left(\frac{n(n+1)}{n^2}\right)^2 - \frac{54}{2} \frac{n(n+1)}{n^2}\right]$$

$$= \lim_{n\to\infty} \left(\frac{81}{4} \left(1 + \frac{1}{n}\right)^2 - \frac{54}{2} \left(1 + \frac{1}{n}\right)\right)$$

$$= \frac{81}{4} - \frac{108}{4} = -\frac{27}{4} = -6.75$$

Properties of integrals: (c is const)

1) $\int_a^b c\,dx = c(b-a)$

2) $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$

3) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

4) $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

6) $\int_a^a f(x)\,dx = 0$

§4.9 Antiderivative

7/21

For what we do next, we need to know what an anti-deri is.

<u>Defn</u>: A func F is the antiderivative of f if $F'(x) = f(x)$.

<u>Thm</u>: The most general antiderivative of   is

$$F(x) + c$$

where c is an arbitrary const.

Ex: Find the antiderivative of the following.

a) $f(x) = \cos x$

b) $f(x) = \frac{1}{x}$

c) $f(x) = x^n \quad n \neq 1$

d) $f(t) = \dfrac{3t^4 - t^3 - 6t^2}{t^4}$

Ex: Find the antiderivative of $f(x) = 1 - 2\sin x + \frac{3}{\sqrt{x}}$ that satisfies $F(0) = 5$.

Ex: If $f''(x) = e^x + \frac{1}{x^2}$, find f.

Ex: Find $f(x)$ when $f''(x) = \sin x + 40x^{2/3}$ and $f'(0) = 2$ and $f(0) = 7$.

So why antideriv's? Cuz they are related to integrals.

§5.3 The Fundamental Thm of Calc

The Fundamental Thm of Calc

Sps f is cont on [a,b].

1) If $g(x) = \int_a^x f(t)\,dt$ then $g'(x) = f(x)$.

come back after example → 2) $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F'(x) = f(x)$.

Ex: Find $g'(x)$ if

a) $g(x) = \int_1^x \ln(1+t^2)\,dx$

b) $g(x) = \int_x^0 \sqrt{1+\sec(t)}\,dt$

c) $g(x) = \int_1^{\sqrt{x}} \frac{z^2}{z^4+1}\,dz$

(Do part 2)

Ex: Evaluate the integral

a) $\int_0^1 x^2\,dx$

b) $\int_1^8 x^{-2/3}\,dx$

c) $\int_{-5}^{5} e\,dt$

d) $\int_0^1 (s+2)(s-3)\,ds$

§5.4 Indefinite Integral

The antiderive is the same as an indefinite integral

$$\boxed{\int f(x)\,dx = F(x) \quad \text{means} \quad F'(x) = f(x)}$$

Ex: $\int \cos x\,dx = \sin x + c$

$\int x^4\,dx = \frac{1}{5}x^5 + c$

$\int \frac{x^2+1}{x}\,dx = \frac{1}{2}x^2 + \ln x + c$

Basics

$\int c f(x)\,dx = c\int f(x)\,dx$ $\qquad$ $\int (f(x)+g(x))\,dx = \int f\,dx + \int g\,dx$

$\int K\,dx = Kx + c$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad n \neq -1$

$\int e^x\,dx = e^x + c$

$\int 1/x\,dx = \ln|x| + c$

$\int \sin x\,dx = -\cos x + c$

$\int \cos x\,dx = \sin x + c$

Ex: Find $\int (10x^4 - 2\sin x)\,dx. = 2x^5 + 2\cos x + c$

Ex: Find $\int \left(\frac{1+t}{t}\right)^2 dt$

Ex: Evaluate $\int_0^{\pi} (5e^x + 3\sin x)\,dx$

Ex: Evaluate $\int_1^2 (4x^3 - 3x^2 - 2x)\,dx$

7/27

§5.5 The substitution Rule

It's like chain rule for integration but ppl get it better.

Ex: $\int 2x\sqrt{1+x^2}\,dx = \int \underbrace{\sqrt{1+x^2}}_{u}\ \underbrace{2x\,dx}_{du} = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + c = \frac{2}{3}(1+x^2)^{3/2} +$

$u = 1+x^2$

$du = 2x\,dx$

Ex: $\int \cos(2x)\,dx = \int \cos(u)\,\frac{1}{2}\,du = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + c$

$= \frac{1}{2}\sin(2x) + c$

$u = 2x$

$du = 2\,dx \Rightarrow \frac{1}{2}\,du = dx$

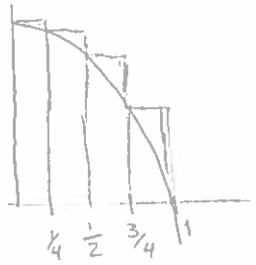
Ex: $\int x^2 e^{x^3}\,dx$

Ex: $\int \sin t\sqrt{1+\cos t}\,dt$

Ex: $\int e^{\cos t}\sin t\,dt$

Ex: $\int (x^2+1)(x^3+3x)^4\,dx$

1. a) Approx the area under the curve $f(x) = 1 - x^2$ from $x = 0$ to $x = 1$ using Riemann summation. Use the left end pt and 4 subintervals of equal spacing. Is this an over or under-estimate?



$$A = \frac{1}{4}(1) + \frac{1}{4}\left(\frac{15}{16}\right) + \frac{1}{4}\left(\frac{3}{4}\right) + \frac{1}{4}\left(\frac{7}{16}\right)$$

$$= \frac{1}{4}\left(\frac{16}{16} + \frac{15}{16} + \frac{12}{16} + \frac{7}{16}\right) = \frac{1}{4}\ \frac{50}{16} = \frac{25}{32}$$

over estimate

b) Repeat a) w/ right end pts

c) Repeat a) w/ mid pts

d) Find the exact area using Riemann summation and limits